New Characterizations of the Pareto Distribution

Zohdy M. Nofal
Department of Statistics, Faculty of Commerce, Benha University, Egypt.
dr.zohdynofal@fcom.bu.edu.eg

Yehia M. El Gebaly
Department of Statistics, Faculty of Commerce, Benha University, Egypt.
yehia1958@yahoo.com

Abstract

Characterization results have great importance in statistics and probability applications. Some characterizations of Pareto of the first kind and Pareto of the second kind distributions are presented by using conditional expectation in terms of their failure (hazard) rate. We also provide two characterization theorems based on the rth truncated moments.

Keywords: Characterization, Failure Rate, Conditional Expectation, Mixture, Pareto of The first kind, Pareto of The second kind Distributions.

1. Introduction

In recent years order statistics and their moments have assumed considerable interest, the moments of order statistics have been tabulated quite extensively for several distributions, for example see Arnold et al. (1992) and David (1981). Many papers dealing with characterization through properties of order statistics are appeared, see for example Khan and Abouammoh (1999), Malik et al. (1988), Lin (1988), Kamps ((1991), (1995)), and Mohie El-Din et al. (1991).

Khan and Abu-Salih (1989) have characterized many well-known continuous probability distributions such as Pareto and power function distributions through conditional expectation of functions of order statistics. Ahsanullah and Raqab (2004) have characterized continuous distributions by conditional expectation of some functions of generalized order statistics. Ahsanullah and Hamedani (2007) characterized beta of the first kind and the power function distribution using 1^{st} order statistics and n^{th} order statistics respectively. Hamedani et al. (2008) characterized certain univariate distributions using truncated moments $X_{(1)}$. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbag (1980), Ahsanullah (1989), Oncel et al. (2005) and Wesolowski and Ahsanullah (2004). Ahsanullah (2009) characterized several univariate distributions by the moments of the $(i+1)^{th}$ $(1 \leq i \geq n)$ order statistic given i^{th} order statistic$= t$. In this paper characterizations of some univariate distributions using the s^{th} moments of the $(r+1)^{th}$ order statistic given r^{th} order statistic $= x$ are given. Afify et al. (2013) characterized the exponential and power function distributions using the sth conditional expectation of order statistics.

Let $X_1, X_{2,}\dots, X_n$ be a random sample of size n from an absolutely continuous distribution with cumulative distribution function $(cdf) F(x)$ and the corresponding probability density function $(pdf) f(x)$. Let $X_{(1)}, X_{(2),}\dots, X_{(n)}$ be the corresponding order statistics. Then the pdf of $X_{(r)}$, the joint pdf of $X_{(r)}$ and $X_{(r+1)}$ and the conditional pdf of $X_{(r+1)}$ given $X_{(r)} = x$ are, respectively, see Arnold et al. (1992).

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}, a < x < b. \quad (1.1)$$

$$f_{X_{(r)},X_{(r+1)}}(x,y) = \frac{n!}{(r-1)!(n-r-1)!} f(x)f(y)[F(x)]^{r-1}[1-F(y)]^{n-r-1}, a < x < y < b. \quad (1.2)$$

$$f_{X_{(r+1)}|X_{(r)}}(y|x) = (n-r)\frac{[1-F(y)]^{n-r-1}}{[1-F(x)]^{n-r}} f(y) \quad (1.3)$$

In section 2, the Pareto of the first kind, Pareto of the second kind distributions are to be characterized through truncated moments of order statistics given by:

$$E\left(X_{(r+1)}^s | X_{(r)} = x\right) = \int_x^\infty y^s f_{X_{(r+1)}|X_{(r)}}(y|x)dy, \qquad s = 1,2,3,\ldots, \quad r = 1,2,\ldots,n-1.$$

2. Characterization Theorems

2.1 Characterization of Pareto of the First Kind Distribution

The pdf and the survival function (sf) of the Pareto distribution of the first type are respectively,

$$f(x) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{-(\beta+1)}, \qquad x \geq \alpha, \qquad \beta > 0, \qquad \alpha > 0.$$

$$\overline{F}(x) = \left(\frac{x}{\alpha}\right)^{-\beta}, \qquad x \geq \alpha, \qquad \beta > 0, \qquad \alpha > 0.$$

Theorem 2.1

Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}(\cdot)$, density function $f(\cdot)$ and Failure (hazard) rate function $h(\cdot)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample of size n from $F(\cdot)$. The random variable X has the Pareto distribution of the first type if and only if

$$E\left(X_{(r+1)}^s | X_{(r)} = x\right) = x^s + \frac{s x^{s+1} h(x)}{\beta^2(n-r) - s\beta}, \; s = 1,2,\ldots, \; r = 1,2,\ldots,n-1, \; h(x) = \frac{\beta}{x}. \quad (2.1)$$

Proof.

(Necessity):

Observe that

$$E\left(X_{(r+1)}^s | X_{(r)} = x\right) = \int_x^\infty y^s f_{X_{(r+1)}|X_{(r)}}(y|x)dy.$$

Using equation (1.3), we obtain

$$E\left(X_{(r+1)}^{s} | X_{(r)} = x\right) = \frac{n-r}{[1-F(x)]^{n-r}} \int_{x}^{\infty} \beta \alpha^{s-1} \left(\frac{y}{\alpha}\right)^{s-1} \left(\left(\frac{y}{\alpha}\right)^{-\beta}\right)^{n-r} dy$$

$$= \frac{n-r}{[1-F(x)]^{n-r}} A. \tag{2.2}$$

Where

$$A = \int_{x}^{\infty} \beta \alpha^{s-1} \left(\frac{y}{\alpha}\right)^{s-1} \left(\left(\frac{y}{\alpha}\right)^{-\beta}\right)^{n-r} dy$$

$$= \frac{\beta x^{s}}{\beta(n-r)-s} \left(\left(\frac{x}{\alpha}\right)^{-\beta}\right)^{n-r} = \frac{\beta x^{s}}{\beta(n-r)-s} [1-F(x)]^{n-r}. \tag{2.3}$$

Substitution equation (2.3) into equation (2.2), we obtain

$$E\left(X_{(r+1)}^{s} | X_{(r)} = x\right) = x^{s} + \frac{s x^{s+1} h(x)}{\beta^{2}(n-r) - s\beta} \tag{2.4}$$

(**Sufficiency):**

Notice that equation (2.4) can be reduced to

$$\int_{x}^{\infty} (n-r) y^{s} f(y) [1-F(y)]^{n-r-1} dy = \left(x^{s} + \frac{s x^{s}}{\beta(n-r)-s}\right) [1-F(x)]^{n-r}. \tag{2.5}$$

Differentiating the both sides of equation (2.5) with respect to x, we obtain

$$-(n-r) x^{s} f(x) \left(\overline{F}(x)\right)^{n-r-1} = -(n-r) x^{s} f(x) \left(\overline{F}(x)\right)^{n-r-1} - \frac{(n-r) s x^{s}}{\beta(n-r)-s} f(x) \left(\overline{F}(x)\right)^{n-r-1} + \left(\frac{\beta(n-r) s x^{s-1}}{\beta(n-r)-s}\right) \left(\overline{F}(x)\right)^{n-r},$$

$$\frac{(n-r) s x^{s} f(x) \left(\overline{F}(x)\right)^{n-r-1}}{\beta(n-r)-s} = \frac{\beta(n-r) s x^{s-1} \left(\overline{F}(x)\right)^{n-r-1}}{\beta(n-r)-s}$$

$$x f(x) = \beta \overline{F}(x),$$

or equivalently

$$\frac{f(x)}{\overline{F}(x)} = \frac{\beta}{x}. \tag{2.6}$$

Integrating the both sides of equation (2.6) with respect to x, we obtain

$$\ln \overline{F}(x) = \ln x^{-\beta} + \ln k, \qquad \text{where } k \text{ is constant,}$$

$$\ln\left(\frac{\overline{F}(x)}{k}\right) = \ln x^{-\beta},$$

$$\frac{d}{dx}\overline{F}(x) = \frac{d}{dx}kx^{-\beta},$$
$$f(x) = k\beta x^{-(\beta+1)}.$$

From the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$. Then $k = \alpha^{\beta}$, hence

$$f(x) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{-(\beta+1)}, \qquad x \geq \alpha, \qquad \beta > 0, \qquad \alpha > 0\,.$$

Which is the pdf of the Pareto distribution of the first type.

This completes the proof.

Remark 1. Specifying $\mathrm{s} = 1$ and $\mathrm{s} = 2$ in equation (2.1) yields the following results

(i) $E\big(X_{(r+1)}|X_{(r)} = x\big) = x + \frac{x}{\beta(n-r)-1}$.

(ii) $E\big(X_{(r+1)}^2|X_{(r)} = x\big) = x^2 + \frac{2x^2}{\beta(n-r)-2}$.

Then,

$$Var\big(X_{(r+1)}|X_{(r)} = x\big) = \frac{(n-r)x^3 h(x)}{(\beta(n-r)-2)(\beta(n-r)-1)^{2}}.$$

Remark 2. Specifying $\mathrm{s} = 1$ in equation (2.1) gives the result of Ahsanullah (2009).

2.2 Characterization of Pareto of the Second Kind Distribution

In the sequel, we shall use the following symbol $m_{(r)}$

$$m_{(r)} = m(m-1)(m-2)\ldots(m-r+1), \qquad m \neq 0, r = 1,2,3,\ldots\;.$$

The pdf and the sf of the Pareto distribution of the second type are respectively,

$$f(x) = \left(\frac{k}{\theta}\right)\left(1 + \frac{x}{\theta}\right)^{-(k+1)}, x > 0, k, \theta > 0.$$
$$\overline{F}(x) = \left(1 + \frac{x}{\theta}\right)^{-k}, x > 0, k, \theta > 0.$$

Theorem 2.2

Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function $\overline{F}(\cdot)$, density function $f(\cdot)$ and Failure (hazard) rate function $h(\cdot)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample of size n from $F(\cdot)$. The random variable X has the Pareto distribution of the second type if and only if

$$E\big(X_{(r+1)}^s|X_{(r)} = x\big) = \sum_{j=0}^{s} \frac{m_{(1)} s!\, x^{s-j}\theta^j \left(1 + \frac{x}{\theta}\right)^j}{(s-j)!\, m_{(j+1)}}, \qquad m = k(n-r),$$
$$r = 1,2,\ldots,n-1,\; s = 1,2,3,\ldots\;. \qquad (2.7)$$

The following two lemmas are used to prove the sufficiency of theorem 2.2. The two lemmas are proved in the appendix.

Lemma 1

$$\frac{d}{dx}\sum_{j=0}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}=\frac{m_{(1)}}{\theta\left(1+\frac{x}{\theta}\right)}\sum_{j=1}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}.$$

Lemma 2

$$\sum_{j=0}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}=x^{s}+\sum_{j=1}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}.$$

Proof.

(Necessity):

Observe that

$$E\left(X_{(r+1)}^{s}|X_{(r)}=x\right)=\int_{x}^{\infty}y^{s}f_{X_{(r+1)}|X_{(r)}}(y|x)dy.$$

Using equation (1.3), we obtain

$$E\left(X_{(r+1)}^{s}|X_{(r)}=x\right)=\frac{n-r}{[1-F(x)]^{n-r}}\int_{x}^{\infty}\frac{ky^{s}}{y+\theta}\left[\left(1+\frac{y}{\theta}\right)^{-k}\right]^{n-r}dy$$

Let $u=y+\theta$, then

$$E\left(X_{(r+1)}^{s}|X_{(r)}=x\right)=\frac{n-r}{[1-F(x)]^{n-r}}\int_{x+\theta}^{\infty}\frac{k}{u}(u-\theta)^{s}\left(\frac{u}{\theta}\right)^{-k(n-r)-1}du$$

$$=\frac{n-r}{[1-F(x)]^{n-r}}A. \tag{2.8}$$

Where

$$A=\int_{x+\theta}^{\infty}\frac{k}{u}(u-\theta)^{s}\left(\frac{u}{\theta}\right)^{-k(n-r)-1}du\,. \tag{2.9}$$

Integrating the right hand side of equation (2.9) by parts *s* times, we obtain

$$A=\frac{[1-F(x)]^{n-r}}{n-r}\left\{x^{s}+\frac{m_{(1)s!x^{s-1}\theta\left(1+\frac{x}{\theta}\right)}}{(s-1)!\,m_{(2)}}+\frac{m_{(1)s!x^{s-2}\theta^{2}\left(1+\frac{x}{\theta}\right)^{2}}}{(s-2)!\,m_{(3)}}+\ldots+\frac{m_{(1)s!\theta^{s}\left(1+\frac{x}{\theta}\right)^{s}}}{0!\,m_{(s+1)}}\right\}. \tag{2.10}$$

Substituting equation (2.10) into equation (2.8), we obtain

$$E\left(X_{(r+1)}^{s} | X_{(r)} = x\right) = \sum_{j=0}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}}. \tag{2.11}$$

(Sufficiency)

Notice that equation (2.11) can be rewritten as follows:

$$\int_{x}^{\infty} (n-r) y^{s} f(y) [1-F(y)]^{n-r-1} dy$$
$$= [1-F(x)]^{n-r} \sum_{j=0}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}}. \tag{2.12}$$

Defferentiating the both sides of equation (2.12) with respect to x, we obtain

$$-(n-r) x^{s} f(x) \left(\overline{F}(x)\right)^{n-r-1}$$
$$= \left(\overline{F}(x)\right)^{n-r} \frac{d}{dx} \sum_{j=0}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}}$$
$$-(n-r) f(x) \left(\overline{F}(x)\right)^{n-r-1} \sum_{j=0}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}} .$$

Using Lemma (1) and Lemma (2) and simplifying, we obtain

$$(n-r) f(x) \sum_{j=1}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}} = \frac{m_{(1)}}{\theta\left(1+\frac{x}{\theta}\right)} \overline{F}(x) \sum_{j=1}^{s} \frac{m_{(1)} s!\, x^{s-j} \theta^{j} \left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\, m_{(j+1)}},$$

$$(n-r) f(x) = \frac{m_{(1)}}{\theta\left(1+\frac{x}{\theta}\right)} \overline{F}(x),$$

or equivalently

$$\frac{f(x)}{\overline{F}(x)} = \frac{m_{(1)}}{\theta(n-r)\left(1+\frac{x}{\theta}\right)}, \text{where } m_{(1)} = k(n-r), \text{hence}$$

$$\frac{f(x)}{\overline{F}(x)} = \frac{k}{\theta\left(1+\frac{x}{\theta}\right)} . \tag{2.13}$$

Integrating the right hand side of equation (2.13) with respect to x, we obtain

$$\int \frac{f(x)}{\overline{F}(x)} dx = \int \frac{k}{\theta\left(1+\frac{x}{\theta}\right)} dx$$

$$\ln\bar{F}(x) = -k\ln\left(1+\frac{x}{\theta}\right) + \ln c, \text{where } c \text{ is constant}$$

$$\ln\left(\frac{\bar{F}(x)}{c}\right) = \ln\left(1+\frac{x}{\theta}\right)^{-k}$$

$$\bar{F}(x) = c\left(1+\frac{x}{\theta}\right)^{-k}.$$

Using the fact that $\bar{F}(0) = 1$. Then $c = 1$.

Hence

$$\bar{F}(x) = \left(1+\frac{x}{\theta}\right)^{-k}, x \geq 0, k, \theta > 0$$

Which is the sf of the Pareto distribution of the second type.

This completes the proof.

Remark 3. Specifying $\mathrm{s} = 1$ and $\mathrm{s} = 2$ in equation (2.7) yields the following results

(i) $E\left(X_{(r+1)}|X_{(r)} = x\right) = x + \dfrac{x+\theta}{m-1}.$

(ii) $E\left(X^2_{(r+1)}|X_{(r)} = x\right) = x^2 + \dfrac{2x(x+\theta)}{m-1} + \dfrac{2(x+\theta)^2}{(m-1)(m-2)}.$

Then,

$$Var\left(X_{(r+1)}|X_{(r)} = x\right) = \frac{2x(x+\theta)}{m-1} + \frac{m(x+\theta)^2}{(m-1)^2(m-2)}.$$

2.3 Characterization of Two Mixture of Pareto Distributions

Theorem 2.3

Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function density function $\bar{F}(\cdot)$, density function $f(\cdot)$, Failure (hazard) rate function$h(\cdot)$, $E\left(X^j\right) = \mu^j$. The random variable X has the Pareto distribution as

$$f(x) = \beta\alpha^{\beta}x^{-\beta-1}, \quad \bar{F}(x) = \alpha^{\beta}x^{-\beta}, \quad x \geq \alpha, \ \alpha, \beta > 0.$$

If and only if

$$E(X^j|X \geq x) = \mu^j \frac{x^{j+1}}{\beta\alpha^j} h(x), \qquad j < \beta.$$

proof.

(Necessity):

Let X has a Pareto distribution with $\alpha, \ \beta > 0$ which is shap and scale parameters. Then

$$\bar{F}(x)E(X^j|X \geq x) = \int_x^{\infty} u^j f(u)du = \int_x^{\infty} \alpha^{\beta}\beta u^{j-\beta-1}du$$

$$= \alpha^{\beta}\beta \frac{u^{j-\beta}}{j-\beta}\Big|_x^{\infty} = -\frac{\beta\alpha^{\beta}}{j-\beta}x^{j-\beta}$$
$$= \frac{x}{\beta-j}f(x).$$

Then

$$E\left(X^j \middle| X \geq x\right) = \mu^j \frac{x^{j+1}}{\beta\alpha^j}h(x). \qquad (2.14)$$

The proof is completed.

(Sufficiency):

Notice that equation (2.14) can be rewritten as

$$\int_x^{\infty} u^j f(u)d\mathrm{u} = \frac{x^{j+1}}{\beta-j}f(x).$$

By differentiating both sides with respect to x, then

$$-x^j f(x) = \frac{x^{j+1}}{\beta-j}f'(x) + \frac{(j+1)x^j}{\beta-j}f(x).$$

By gathering the simillar terms, then

$$-x^{j+1}f'(x) = (\beta+1)x^j f(x),$$

or

$$\frac{f'(x)}{f(x)} = -\frac{(\beta+1)}{x^j}.$$

Integrating both sides, with respect to x, we get

$$f(x) = kx^{-\beta-1}.$$

Using the fact that $\int_{\alpha}^{\infty} f(x)dx = 1$. Then $k = \beta\alpha^{\beta}$. Thus, we have

$$f(x) = \beta\alpha^{\beta}x^{-\beta-1}.$$

Which is the density function of the Pareto distribution.

Corollary 4. If $j = 1$ then the first truncated moment will be as

$$E(X|X \geq x) = \frac{x^2}{\beta-1}h(x).$$

Corollary 5. The mean residual life can be written as

$$E((X-\mu)|X \geq x) = \frac{x^2}{\beta-1}[h(x) - \alpha\beta].$$

Corollary 6. if $j = 2$ then the second truncated moment will be as

$$E(X^2|X \geq x) = \frac{x^3}{\beta - 2} h(x).$$

Corollary 7. Using corollary (4) and (6), we can compute the truncated variance as

$$V(X|X \geq x) = \frac{x^3}{\beta - 2} h(x) - \frac{x^2}{[\beta - 1]^2} [h(x)]^2.$$

Theorem 2.4

Let X be a nonnegative continuous random variable with distribution function $F(\cdot)$, survival (reliability) function density function $\overline{F}(\cdot)$, density function $f(\cdot)$, reversed Failure (hazard) rate function $\eta(\cdot)$. Then X has the Pareto distribution as

$$f(x) = \beta\alpha^{\beta} x^{-\beta-1}, \quad \overline{F}(x) = \alpha^{\beta} x^{-\beta}, \quad x \geq \alpha, \quad \alpha > 0, \quad \beta > 0,$$

if and only if

$$E\left(X^j \middle| X \leq x\right) = \mu^j + \frac{x}{\beta - j}\left(\alpha^j - x^j\right)\eta(x).$$

Proof.

(Necessity):

Let X has a Pareto distribution with parameters $\alpha > 0$, $\beta > 0$. Then,

$$F(x)E(X^j|X \leq x) = \int_{\alpha}^{x} u^j f(u) d\text{u} = \int_{\alpha}^{x} \alpha^{\beta}\beta u^{j-\beta-1} du$$
$$= \left(\alpha^{\beta}\beta \frac{u^{j-\beta}}{j-\beta} |_{\alpha}^{x}\right) = \frac{\beta\alpha^{\beta}}{j-\beta}\left[x^{j-\beta} - \alpha^{j-\beta}\right].$$

After adding some terms and rearranging the terms, we get

$$F(x)E(X^j|X \leq x) = \mu^j F(x) + \frac{x}{\beta - j} f(x)\left[\alpha^j - x^j\right] .$$

Then

$$E(X^j|X \leq x) = \mu^j + \frac{x}{\beta - j}\eta(x)\left[\alpha^j - x^j\right].$$

The proof is completed.

(Sufficiency):

Notice that the last equation can be rewritten as

$$(\beta - j)\int_{\alpha}^{x} u^j f(u) d\text{u} = \beta\alpha^j F(x) + x f(x)\left[\alpha^j - x^j\right] .$$

By differentiating the both sides with respect to x then,

$$(\beta - j)x^j f(x) = \beta\alpha^j f(x) + x f'(x)\left[\alpha^j - x^j\right] + f(x)\left[\alpha^j - (j+1)x^j\right].$$

By gathering the simillar terms, then

$$-xf'(x) = -(\beta+1)f(x),$$

or

$$\frac{f'(x)}{f(x)} = -\frac{(\beta+1)}{x^j}.$$

Integrating the both sides, with respect to x, we get $f(x) = kx^{-\beta-1}$. Using the fact that

$$\int_{\alpha}^{\infty} f(x)dx = 1.$$

Then $k = \beta\alpha^{\beta}$ and $f(x) = \beta\alpha^{\beta}x^{-\beta-1}$ which is the density function of the Pareto distribution.

Corollary 8. If $j = 1$, then the first truncated moment will be as

$$E(X|X \le x) = \mu + \frac{x}{\beta-1}(\alpha - x)\eta(x).$$

Corollary 9. The mean inactivity time can be written as

$$E((\mu - X)|X \ge x) = -\frac{x}{\beta-1}(\alpha - x)\eta(x).$$

Corollary 10. If $j = 2$, then the second truncated moment will be as

$$E(X^2|X \le x) = \mu^2 + \frac{x}{\beta-2}\eta(x)[\alpha^2 - x^2].$$

Corollary 11. Using corollary (8) and (10), we can compute the truncated variance as

$$V(X|X \le x) = \frac{x(\alpha-x)\eta(x)}{(\beta-2)(\beta-1)^2}[(\alpha+x)(\beta-1)^2 - 2\alpha\beta(\beta-2)] - \frac{x^2(\alpha-x)^2\eta^2(x)}{(\beta-1)^2}.$$

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Appendix

Proof of Lemma 1

$$\frac{d}{dx}\sum_{j=0}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}=$$

$$\left[sx^{s-1}+\frac{m_{(1)s!x^{s-1}}}{(s-1)!\,m_{(2)}}\right]+\left[\frac{m_{(1)s!x^{s-2}\theta\left(1+\frac{x}{\theta}\right)}}{(s-2)!\,m_{(2)}}+\frac{2m_{(1)s!x^{s-2}\theta\left(1+\frac{x}{\theta}\right)^{s-1}}}{(s-3)!\,m_{(3)}}\right]$$

$$+\ldots+\left[\frac{m_{(1)s!\theta^{s-1}\left(1+\frac{x}{\theta}\right)}}{0!\,m_{(s)}}+\frac{sm_{(1)s!\theta^{s-1}\left(1+\frac{x}{\theta}\right)^{s-1}}}{0!\,m_{(s+1)}}\right]$$

$$=m_{(1)}\frac{m_{(1)s!x^{s-1}}}{(s-1)!\,m_{(2)}}+m(1)\frac{m_{(1)s!x^{s-2}\theta\left(1+\frac{x}{\theta}\right)}}{(s-2)!\,m_{(3)}}$$

$$+\ldots+m_{(1)}\frac{m_{(1)s!\theta^{s-1}\left(1+\frac{x}{\theta}\right)^{s-1}}}{0!\,m_{(s+1)}}$$

$$=\frac{m_{(1)}}{\theta\left(1+\frac{x}{\theta}\right)}\left[\frac{m_{(1)s!x^{s-1}}\theta\left(1+\frac{x}{\theta}\right)}{(s-1)!\,m_{(2)}}+\frac{m_{(1)s!x^{s-2}\theta^{2}\left(1+\frac{x}{\theta}\right)^{2}}}{(s-2)!\,m_{(3)}}\right.$$

$$\left.+\ldots+\frac{m_{(1)s!\theta^{s}\left(1+\frac{x}{\theta}\right)^{s}}}{0!\,m_{(s+1)}}\right]$$

$$=\frac{m_{(1)}}{\theta\left(1+\frac{x}{\theta}\right)}\sum_{j=1}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}.$$

Proof of Lemma 2

$$\sum_{j=0}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}=x^{s}+\left[\frac{m_{(1)s!x^{s-1}}\theta\left(1+\frac{x}{\theta}\right)}{(s-1)!\,m_{(2)}}+\frac{m_{(1)s!x^{s-2}}\theta^{2}\left(1+\frac{x}{\theta}\right)^{2}}{(s-2)!\,m_{(3)}}\right.$$

$$\left.+\cdots+\frac{m_{(1)s!}\theta^{s}\left(1+\frac{x}{\theta}\right)^{s}}{0!\,m_{(s+1)}}\right]$$

$$=x^{s}+\sum_{j=1}^{s}\frac{m_{(1)}s!\,x^{s-j}\theta^{j}\left(1+\frac{x}{\theta}\right)^{j}}{(s-j)!\,m_{(j+1)}}.$$